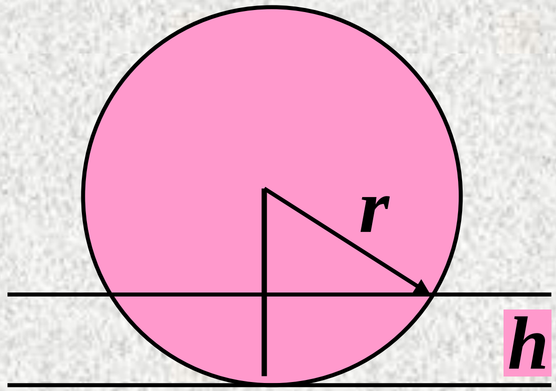
第七章 非线性方程(组)的数值解法

引例1（P37 实验四）一半径为r密度为ρ的球体浸在水中，求浸在水中的深度h.



解　球的质量: $M_b = \frac{4}{3}\pi r^3 \rho$

排开水的质量: $M_w = \int_0^h \pi[r^2 - (r-x)^2]dx$

$$= \frac{\pi}{3}[3r - h]h^2$$

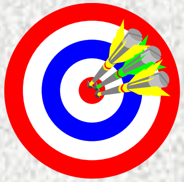
由Archimedes定律：$M_w = M_b$

即需求解: $\frac{\pi}{3}[h^3 - 3h^2 + 4r^3\rho] = 0$

引例2 （P38-6）开普勒（Kepler）方程

$$F(x,y)=y-x-\varepsilon\sin y=0(0<\varepsilon<1)$$

它确定了隐函数 $y=f(x)$.（可以证明 $\forall x$, 有唯一的 y)



求 $f(x)=0$ 的根

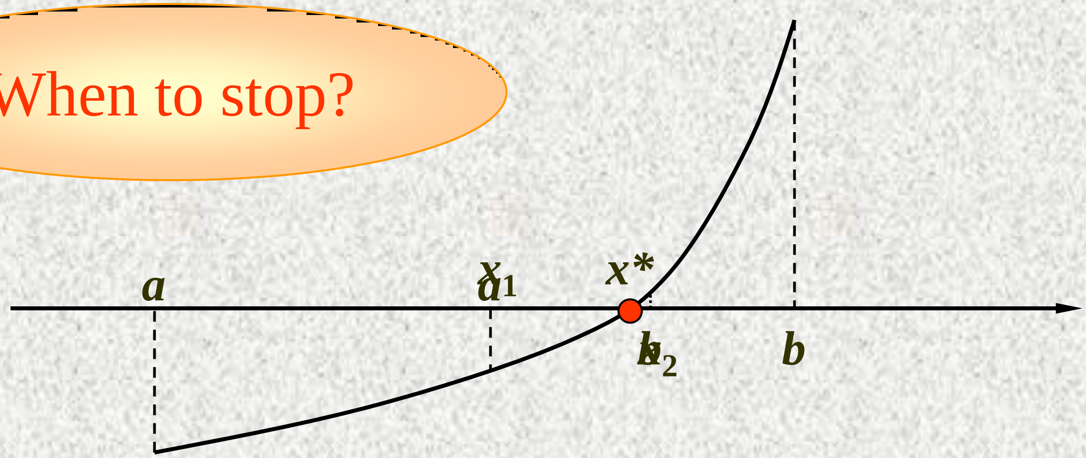
§1 二 分 法

1 根的隔离与隔根区间

2 二分法

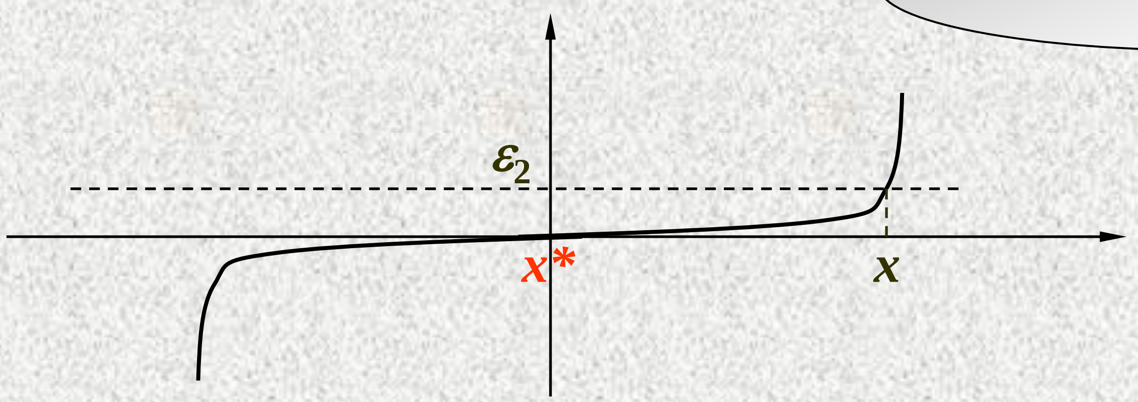
原理：若 $f \in C[a, b]$，且 $f(a) \cdot f(b) < 0$，则 f 在 (a, b) 上必有一根。

When to stop?



$$\left|x_{k+1}-x_k\right|<\varepsilon_1 \quad 或 \quad \left|f\left(x_{k+1}\right)\right|<\varepsilon_2$$

能不能保证 x 的精度?



误差分析 第1步产生的 $x_1=\dfrac{a+b}{2}$ 有误差 $|x_1-x^*|\le\dfrac{b-a}{2}$

第 k 步产生的 x_k 有误差 $|x_k-x^*|\le\dfrac{b-a}{2^k}$

对于给定的精度 ε,可估计二分法所需的步数 k:

$$\frac{b-a}{2^k}<\varepsilon \quad\Rightarrow\quad k>\frac{[\ln(b-a)-\ln\varepsilon]}{\ln 2}$$

①简单;对 $f(x)$ 要求不高(只要连续即可).

②无法求复根，收敛慢

注：用二分法求根，最好先给出 $f(x)$ 草图以确定根的大概位置。或用搜索程序，将$[a, b]$分为若干小区间，对每一个满足 $f(a_k)f(b_k)<0$ 的区间调用二分法程序，可找出区间$[a, b]$内的多个根，且不必要求 $f(a)f(b)<0$ 。

§ 2 不动点迭代法

一 问题 设$f(x)$连续，求方程$f(x)=0$的根。

二 基本思想 转化为不动点求解

$$f(x)=0 \xLeftrightarrow{\text{等价变形}} x=\varphi(x)$$

如果$\exists x^*$,使$x^*=\varphi(x^*)$,则称x^*为$\varphi(x)$的一个不动点.

三 基本思想与迭代格式

任取x_0(一般$\neq x^*$)，构造迭代:

$$x_1=\varphi(x_0),\quad x_2=\varphi(x_1),\quad \cdots \quad x_{k+1}=\varphi(x_k)$$

得到一序列$\{x_k\}$. 如果$\{x_k\}$的极限存在,

设$\lim\limits_{k\to\infty} x_k = x^*$,则显然有

$$x^*=\varphi(x^*)$$

即x^*是φ的不动点，此时称迭代$x_{k+1}=\varphi(x_k)$收敛.

$$\boxed{x^*=\lim_{k\to\infty}x_k=\lim_{k\to\infty}\varphi(x_{k-1})=\varphi(\lim_{k\to\infty}x_{k-1})=\varphi(x^*)}$$

基本迭代格式 $x_{k+1}=\varphi(x_k)$

迭代函数 $\varphi(x)$

迭代收敛 $\lim\limits_{k\to\infty} x_k = x^*.$

迭代发散 $\lim\limits_{k\to\infty} x_k$不存在

例1 建立迭代格式求$x^3 - x^2 - 1 = 0$在[1.3,1.6]内的根.

✍方案1: $x^3 - x^2 - 1 = 0$

$$\Leftrightarrow x^3 = x^2 + 1 \quad \Leftrightarrow x = \sqrt[3]{(1 + x^2)}$$

$\therefore$ 建立迭代格式: $x_{k+1} = (1 + x_k^2)^{1/3}$

取迭代初值$x_0 = 1.3$,可算得:

$$k = 0: \quad x_1 = (1 + x_0^2)^{1/3} = (1 + 1.3^2)^{1/3} = 1.3907$$

$$k = 1: \quad x_2 = (1 + x_1^2)^{1/3} = (1 + 1.3907^2)^{1/3} = 1.4316$$

$$k = 2: \quad x_3 = (1 + x_2^2)^{1/3} = (1 + 1.4316^2)^{1/3} = 1.4501$$

$$\cdots \quad \cdots$$

$$k = 6: \quad \boxed{x_7 = \cdots = 1.4649}$$

方案2: $x^3 - x^2 - 1 = 0$

$$\Leftrightarrow x^3 = x^2 + 1$$

$$\Leftrightarrow x = 1 + \frac{1}{x^2}$$

迭代格式: $x_{k+1} = 1 + \dfrac{1}{x_k^2}$

方案3: $x^3 - x^2 - 1 = 0$

$$\Leftrightarrow x^2(x-1) = 1$$

$$\Leftrightarrow x = \frac{1}{\sqrt{(x-1)}}$$

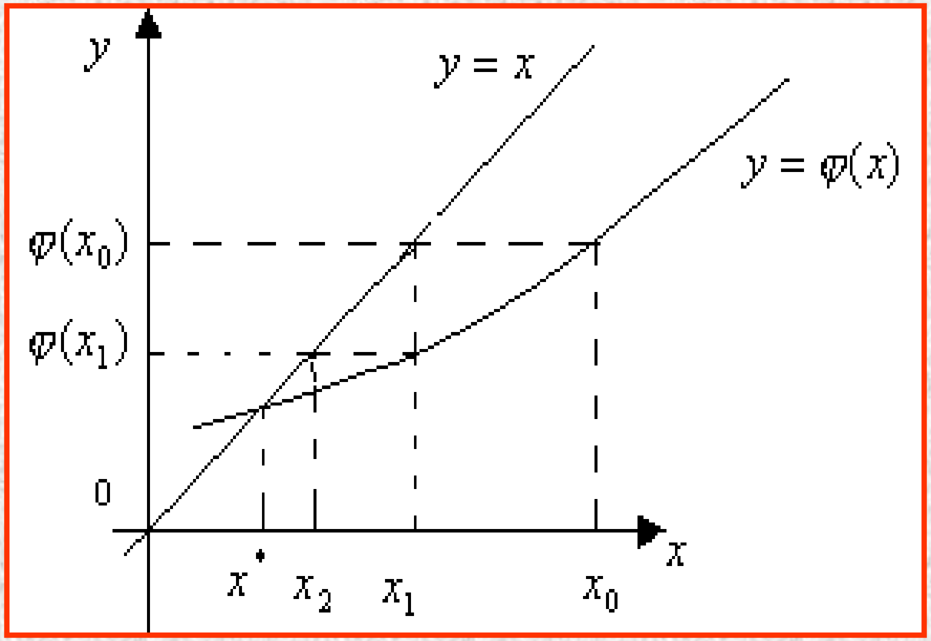
迭代格式: $x_{k+1} = \dfrac{1}{(x_k - 1)^{1/2}}$

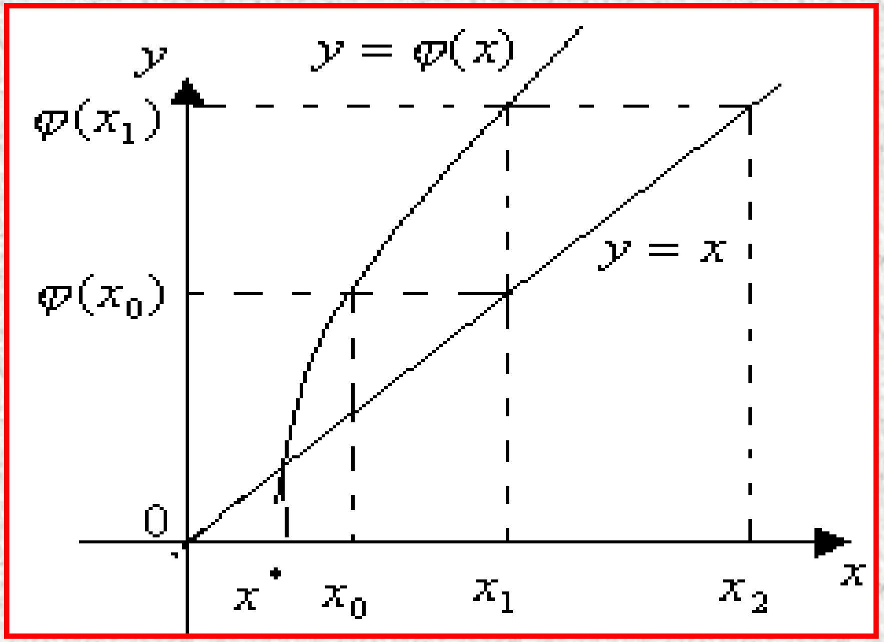
取$x_0 = 1.3$,计算结果为:

k	格式2	格式3
1	1.5917	1.8257
2	1.3947	1.0047
3	1.5149	3.1549
4	1.4362	0.6812
…		
13	1.4653	

四 几何意义

求$x=\varphi(x)$的根 $\Leftrightarrow$ 求$\begin{cases} y=x \\ y=\varphi(x) \end{cases}$交点的横坐标.





收敛情形

发散情形

五 收敛条件和误差估计

定理1(P156,不动点存在唯一定理)

设$\varphi'(x)$在$[a,b]$上存在且满足:

(1)对$\forall x \in [a,b]$,有$a \le \varphi(x) \le b$,

(2)$\exists L: 0 \le L < 1$,对$\forall x \in [a,b]$有$|\varphi'(x)| \le L$.

则φ在$[a,b]$上存在唯一不动点$x^*(x^* = \varphi(x^*))$.

证明:1)存在性

令$f(x) = x - \varphi(x)$,则由$\varphi'(x)$存在知$f(x) \in C[a,b]$

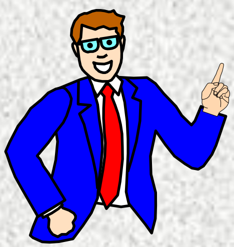
由(1):$f(a) = a - \varphi(a) \le 0$, $f(b) = b - \varphi(b) \ge 0$

$\therefore \exists x^*$,使$f(x^*) = 0$, $\boxed{即x^* = \varphi(x^*).}$

2)唯一性 设$x = \varphi(x)$有另一根α,即$\alpha = \varphi(\alpha)$,则

$$|x^* - \alpha| = |\varphi(x^*) - \varphi(\alpha)| = |\varphi'(\xi)| \cdot |x^* - \alpha|$$

$$\le L|x^* - \alpha| < |x^* - \alpha| \qquad \boxed{\therefore x^* = \alpha}$$



对定理条件的理解:

条件1: $\forall x \in [a,b]$,有$a \le \varphi(x) \le b$,保证解的存在性;

条件2: $|\varphi'(x)| \le L$

$\Rightarrow \forall x, y \in [a,b],$

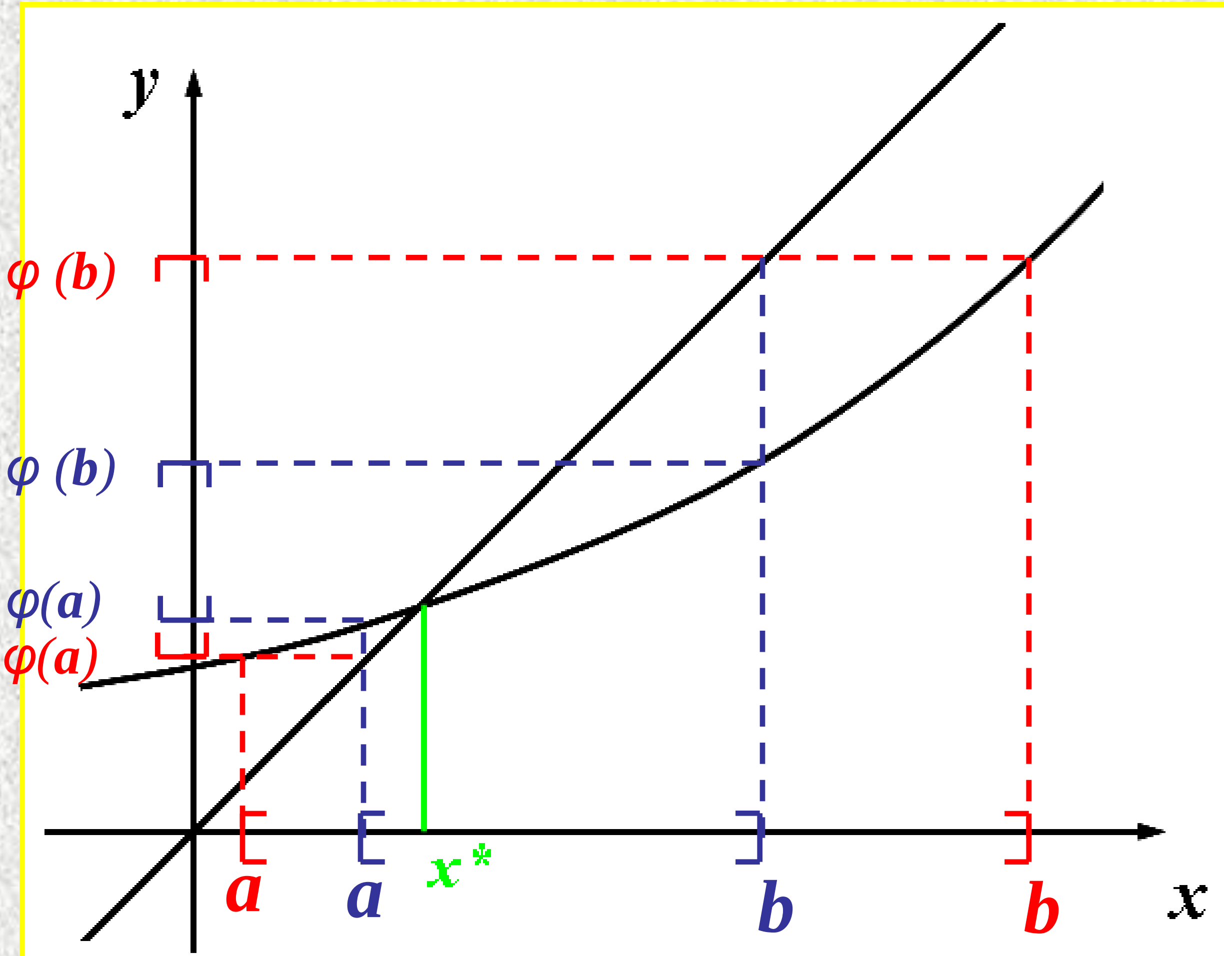
$$|\varphi(x) - \varphi(y)|$$

$$= |\varphi'(\xi)| \cdot |x - y|$$

$$\le L|x - y|$$

即φ为压缩映射。

L:压缩因子



定理 迭代的收敛性及误差估计

设$\varphi'(x)$在$[a,b]$上存在且满足:

(1)对$\forall x\in[a,b]$,有$a\le\varphi(x)\le b$,

(2)$\exists L: 0\le L<1$,对$\forall x\in[a,b]$有$|\varphi'(x)|\le L$.

则对$\forall x_0\in[a,b]$,迭代$x_{k+1}=\varphi(x_k)$收敛于φ的唯一不动点x^*，且有如下误差估计式:

$$|x_k-x^*|\le\frac{L}{1-L}|x_k-x_{k-1}|$$

$$|x_k-x^*|\le\frac{L^k}{1-L}|x_1-x_0|$$

反之，若对$\forall x\in(a,b)$有$|\varphi'(x)|\ge 1$，则迭代发散.

证明 1)收敛性

$$|x_k - x^*| = |\varphi(x_{k-1}) - \varphi(x^*)| = |\varphi'(\xi)| \cdot |x_{k-1} - x^*|$$

$$\le L|x_{k-1} - x^*| \le \cdots \le L^k |x_0 - x^*| \to 0$$

2)误差估计式

$$|x_k - x^*| \le L|x_{k-1} - x^*| = L[|x_{k-1} \boxed{- x_k + x_k} - x^*|]$$

$$\le L[|x_k - x^*| + |x_k - x_{k-1}|]$$

$$\Rightarrow |x_k - x^*| \le \frac{L}{1-L}|x_k - x_{k-1}|$$

$$|x_k - x_{k-1}| = |\varphi(x_{k-1}) - \varphi(x_{k-2})|$$

$$\le L|x_{k-1} - x_{k-2}| \le \cdots \le L^{k-1}|x_1 - x_0|$$

代入上式结论得: $|x_k - x^*| \le \dfrac{L^k}{1-L}|x_1 - x_0|$

3)若 $|\varphi'(x)| \ge 1$, 则

$$|x_{k+1} - x^*| = |\varphi(x_k) - \varphi(x^*)| = |\varphi'(\xi)| \cdot |x_k - x^*| \ge |x_k - x^*|$$

所以发散。

注：1、收敛快慢与L的大小有关：

L越小，收敛越快；L越大，收敛越慢

2、事后误差估计：

$$\left|x^{*}-x_{k}\right| \leq \frac{L}{1-L}\left|x_{k}-x_{k-1}\right|$$

事先误差估计：

$$\left|x_{k}-x^{*}\right| \leq \frac{L^{k}}{1-L}\left|x_{1}-x_{0}\right|$$

3、程序终止条件：$\left|x_{k}-x_{k-1}\right|<\varepsilon$

定理2 （局部收敛性）

设x^*是$x=\varphi(x)$的根，$\varphi'(x)$在x^*连续且$|\varphi'(x^*)|<1$,则存在x^*的某邻域$U(x^*)$,使$\forall x_0\in U(x^*)$,迭代$x_{k+1}=\varphi(x_k)$都收敛.

证明 $\because \varphi'(x)$在x^*连续

$\therefore \exists\delta>0, 0<L<1$,当$x\in U(x^*,\delta)$时, $|\varphi'(x)|\le L<1$

$\therefore$ 对$\forall x\in U(x^*,\delta)$,有

$$|\varphi(x)-x^*|=|\varphi(x)-\varphi(x^*)|\le L|x-x^*|<|x-x^*|$$

即$\varphi(x)\in U(x^*,\delta)$.

$\therefore$ 由定理1可知结论成立.

例2 讨论例1中求 $x^3-x^2-1=0$ 各方法的敛散性

✍方案1: $x_{k+1}=(1+x_k^2)^{1/3}$

$$\because \varphi(x)=(1+x^2)^{1/3}$$

$$\therefore |\varphi'(x)|=\left|\frac{2x}{3(1+x^2)^{2/3}}\right|$$

又 $\because x\in[1.3,1.6]$

$$\therefore |\varphi'(x)|\le 2\times\frac{1.6}{[3(1+1.3)]^{2/3}}\approx 0.5515<1$$

$\therefore$ 迭代收敛

✍方案 2： $x_{k+1} = 1 + \frac{1}{x_k^2}$

$$\because \varphi(x) = 1 + \frac{1}{x^2} \quad \therefore |\varphi'(x)| = \left|\frac{-2}{x^3}\right|$$

又 $\because x \in [1.3, 1.6]$

$$\therefore |\varphi'(x)| \le \frac{2}{1.3^3} = 0.901 \quad \therefore$$ 迭代收敛

✍方案 3： $x_{k+1} = \frac{1}{(x_k - 1)^{1/2}} \quad \because \varphi(x) = \frac{1}{\sqrt{x-1}}$

$$|\varphi'(x)| = \left|\frac{-1}{2(x-1)^{3/2}}\right| > \frac{1}{2(1.6-1)^{2/3}} \approx 1.07582 > 1$$

所以发散

例3 对函数$f(x)$,设对$\forall x$, $f'(x)$存在且$0<m\le f'(x)\le M$.

证明：对任意的$0<\lambda<\dfrac{2}{M}$,迭代过程$x_{k+1}=x_k-\lambda f(x_k)$

均收敛于$f(x)=0$的根x^*.

证明 $\quad f(x)=0 \Leftrightarrow x=x+\lambda f(x)$

$$\because x_{k+1}=x_k-\lambda f(x_k)$$

迭代函数为 $\varphi(x)=x-\lambda f(x)$, $\varphi'(x)=1-\lambda f'(x)$

又 $\because 0<m\le f'(x)\le M$, $\quad 0<\lambda<\dfrac{2}{M}$

$$\therefore 0<\lambda m\le \lambda f'(x)\le \lambda M<2$$

$$-2<-\lambda f'(x)<0,\ -1<1-\lambda f'(x)<1$$

即 $\quad |1-\lambda f'(x)|<1$

$\therefore$ 迭代$x_{k+1}=x_k-\lambda f(x_k)$收敛于$f(x)=0$的根$x^*$

六 收敛速度

1 定义

设$x_{k+1}=\varphi(x_k)$收敛,如果迭代误差$e_k=x^*-x_k$满足

$$\boxed{\lim_{k\to\infty}\frac{e_{k+1}}{(e_k)^p}=c.\ \ (c\neq 0,\text{常数})}$$

则称迭代是p阶收敛的.

特别地：$p=1$时,又叫线性收敛;

$p=2$时,叫平方收敛.

例4 设$a>0$,证明:迭代公式$x_{k+1}=\dfrac{x_k(x_k^2+3a)}{3x_k^2+a}$是计算$\sqrt{a}$的三阶方法.

证明　令$\varphi(x)=\dfrac{x(x^2+3a)}{3x^2+a}$，则:

$$\varphi'(x)=\frac{(3x^2+3a)(3x^2+a)-x(x^2+3a)6x}{(3x^2+3a)^2}=\frac{3(x^2-a)^2}{(3x^2+3a)^2}$$

$$\therefore|\varphi'(x)|\le\left(\frac{\sqrt{3}x^2}{3x^2+3a}\right)^2\le\left(\frac{\sqrt{3}x^2}{3x^2}\right)^2=\frac{1}{3}<1,\text{ 即迭代收敛.}$$

设$\lim\limits_{k\to\infty}x_k=l$，　则$l=l(l^2+3a)/(3l^2+a)$，　解得$l=0,\pm\sqrt{a}$,

由题意知$l=\sqrt{a}$.　即$\lim\limits_{k\to\infty}x_k=\sqrt{a}$

$$\text{又}\because\lim_{k\to\infty}\frac{\sqrt{a}-x_{k+1}}{(\sqrt{a}-x_k)^3}=\lim_{k\to\infty}\frac{\sqrt{a}-\dfrac{x_k(x_k^2+3a)}{3x_k^2+a}}{(\sqrt{a}-x_k)^3}=\lim_{k\to\infty}\frac{(\sqrt{a}-x_k)^3}{(\sqrt{a}-x_k)^3(3x_k^2+a)}$$

$$=\lim_{k\to\infty}\frac{1}{3x_k^2+a}=\frac{1}{4a}\neq0$$

所以是三阶方法.

2 定理4(收敛速度判定定理)

对$x_{k+1}=\varphi(x_k)$,设$\varphi^{(p)}(x)$在根x^*的某邻域内连续,则

(1) 若$0<|\varphi'(x^*)|<1$,则在x^*邻近为一阶收敛;

(2)如果$\varphi'(x^*)=\varphi''(x^*)=\cdots=\varphi^{(p-1)}(x^*)=0$,

而$\varphi^{(p)}(x^*)\neq 0$, 则在 x^*邻近是 p 阶收敛.

证明： 1) $\because 0<|\varphi'(x^*)|<1$

$\therefore \exists$邻域R,使$x\in R$时,$|\varphi'(x)|\le L<1$

$\therefore$ 迭代格式局部收敛

又$\because x_{k+1}-x^* = \varphi(x_k)-\varphi(x^*) = \varphi'(\xi)(x_k-x^*)$

$$\therefore \lim_{k\to\infty}\frac{e_{k+1}}{(e_k)} = \lim_{k\to\infty}\frac{x_{k+1}-x^*}{x_k-x^*}$$

$$= \lim_{k\to\infty}\varphi'(\xi)$$

$$= \varphi'(x^*)\neq 0$$

即迭代过程为线性收敛

(2) 若$\varphi'(x^*)=\varphi''(x^*)=\cdots=\varphi^{(p-1)}(x^*)=0,\ \varphi^{(p)}(x^*)\neq 0$

将$\varphi(x_k)$在 x^*处泰勒展开:

$$\varphi(x_k)=\varphi(x^*)+\frac{1}{p!}\varphi^{(p)}(\xi)(x_k-x^*)^p,\ \ \xi\in(x_k,x^*)$$

$$\therefore e_{k+1}=x_{k+1}-x^*=\varphi(x_k)-\varphi(x^*)$$

$$=\frac{1}{p!}\varphi^{(p)}(\xi)(x_k-x^*)^p=\frac{1}{p!}\varphi^{(p)}(\xi)(e_k)^p$$

$$\therefore \lim_{k\to\infty}\frac{e_{k+1}}{(e_k)^p}=\lim_{k\to\infty}\frac{1}{p!}\varphi^{(p)}(\xi)\overset{k\to\infty}{=\!=\!=}\frac{1}{p!}\varphi^{(p)}(x^*)\neq 0$$

所以,迭代过程为p阶收敛.

例4 设$a>0$,再证：迭代公式$x_{k+1}=\dfrac{x_k(x_k^2+3a)}{3x_k^2+a}$是计算$\sqrt{a}$的三阶方法.

[分析] 记$x^*=\sqrt{a}$, 往证：$\varphi(x^*)=x^*$, $\varphi'(x^*)=0$, $\varphi''(x^*)=0$, $\varphi'''(x^*)\neq 0$

证明 $\varphi(x)=\dfrac{x(x^2+3a)}{3x^2+a}$ $\quad\because \varphi(\sqrt{a})=\dfrac{\sqrt{a}((\sqrt{a})^2+3a)}{3(\sqrt{a})^2+a}=\sqrt{a}$

$\therefore \sqrt{a}$是$x=\varphi(x)$的根

将$\varphi(x)(3x^2+a)-x(x^2+3a)=0$两边对$x$求导,得

$(3x^2+a)\varphi'(x)+6x\varphi(x)-3x^2-3a=0$, 令$x=\sqrt{a}$, 得$\varphi'(\sqrt{a})=0$

上式两端再对x求导,得

$(3x^2+a)\varphi''(x)+12x\varphi'(x)+6\varphi(x)-6x=0$, 令$x=\sqrt{a}$, 得$\varphi''(\sqrt{a})=0$

两端再对x求导,得

$$(3x^2+a)\varphi'''(x)+18x\varphi''(x)+18\varphi'(x)-6=0$$

令$x=\sqrt{a}$, 得 $\varphi'''(\sqrt{a})=\dfrac{3}{2a}\neq 0$, 所以是三阶方法.

七 迭代过程加速方法–Aitken加速法

记 $\boxed{\overline{x_{k+1}} = \varphi(x_k)}$ (1)

$\boxed{\tilde{x}_{k+1} = \varphi(\overline{x_{k+1}})}$ (2)

则 $x^* - \overline{x_{k+1}} = \varphi(x^*) - \varphi(x_k) = \varphi'(\xi)(x^* - x_k),\ \xi \in [x^*, x_k]$

$x^* - \tilde{x}_{k+1} = \varphi'(\eta)(x^* - \overline{x_{k+1}}), \eta \in [x^*, \overline{x_{k+1}}]$

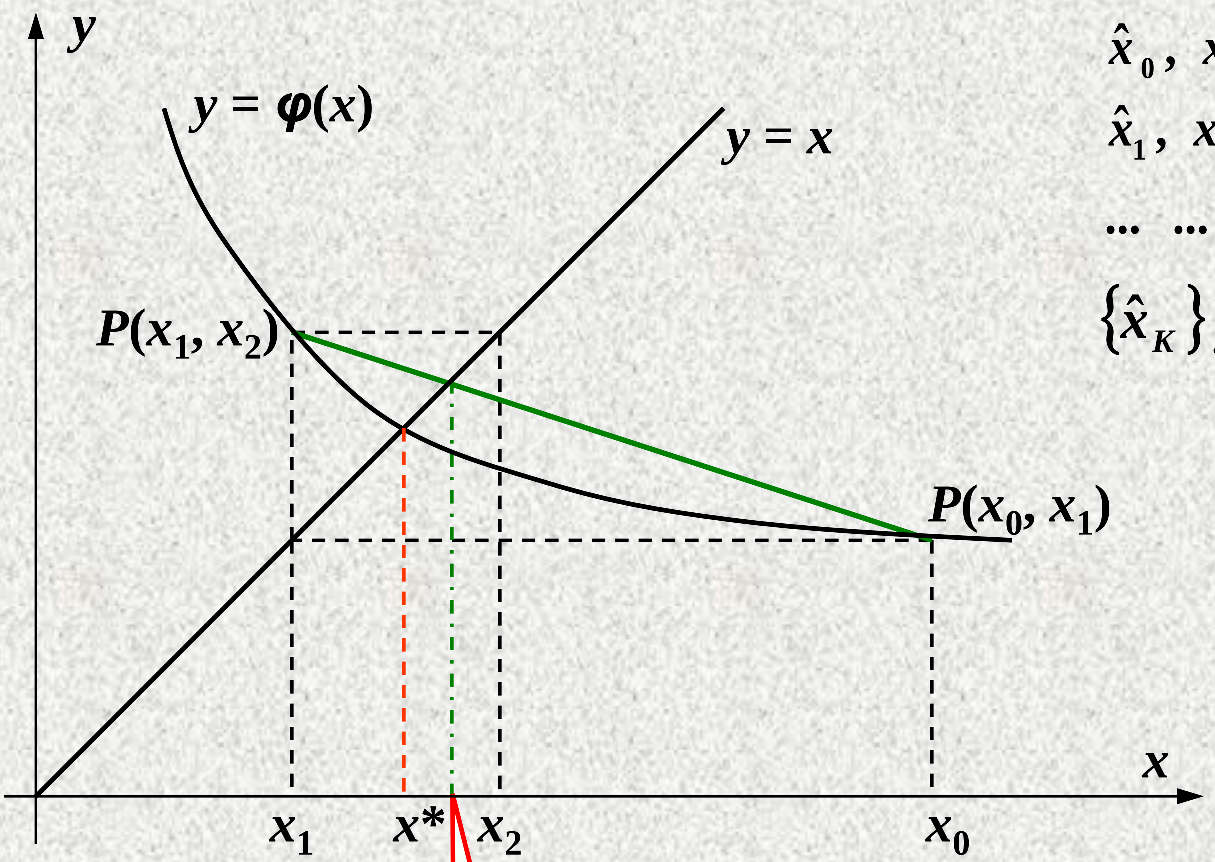
$$\left.\begin{array}{l} \because \xi \approx \eta \\ \therefore \varphi'(\xi) \approx \varphi'(\eta) \end{array}\right\} \therefore \frac{x^* - \overline{x_{k+1}}}{x^* - \tilde{x}_{k+1}} \approx \frac{x^* - x_k}{x^* - \overline{x_{k+1}}}$$

整理得: $x^* - \tilde{x}_{k+1} \approx -\dfrac{(\tilde{x}_{k+1} - \overline{x_{k+1}})^2}{\tilde{x}_{k+1} - 2\overline{x_{k+1}} + x_k}$

令 $\boxed{x_{k+1} = \tilde{x}_{k+1} - \dfrac{(\tilde{x}_{k+1} - \overline{x_{k+1}})^2}{\tilde{x}_{k+1} - 2\overline{x_{k+1}} + x_k}}$ (3)

称(1)(2)(3)为Aitken加速法.

几何意义:



$$\boxed{\hat{x}} = x_0 - \frac{(x_1 - x_0)^2}{x_0 - 2x_1 + x_2}$$

一般地，有：$\hat{x}_K = x_K - \dfrac{(x_{K+1} - x_K)^2}{x_K - 2x_{K+1} + x_{K+2}}$

x_0，$x_1 = \phi(x_0)$，$x_2 = \phi(x_1)$，

$\hat{x}_0$，$x_3 = \phi(x_2)$，

$\hat{x}_1$，$x_4 = \phi(x_3)$，

… …

$\{\hat{x}_K\}$比$\{x_K\}$收敛得略快。

例5 求$f(x)=x^3-x-1=0$的唯一正根 $\varepsilon=0.005$.

✍方案1: $x=\sqrt[3]{x+1}$

则 $x_{k+1}=\sqrt[3]{x_k+1} \quad (k=0,1,2,\cdots)$

取 $x_0=1.5$, 则 $x_1=1.35721$, $x_2=1.33086$,

$x_3=1.32588$

$\because \left|x_3-x_2\right|<\varepsilon=0.005$

$\therefore x^*\approx x_3=1.32588$

✍方案2: $x = x^3 - 1$

$$x_{k+1} = x_k^3 - 1$$

取 $x_0 = 1.5$,则

$x_1 = 2.375$,

$x_2 = 12.3965$,

$\cdots$

$\to \infty$, 发散.

✍方案3: 对法二Aitken加速

$$\bar{x}_{k+1} = x_k^3 - 1$$

$$\tilde{x}_{k+1} = \bar{x}_{k+1}^3 - 1$$

$$x_{k+1} = \tilde{x}_{k+1} - \frac{(\tilde{x}_{k+1} - \bar{x}_{k+1})^2}{\tilde{x}_{k+1} - 2\bar{x}_{k+1} + x_k}$$

Aitken加速计算结果:

k	$\bar{x}_k$	$\tilde{x}_k$	x_k
0	None	None	1.5
1	2.375	12.3965	1.41629
2	1.18409	…	1.35565
3	1.49140	…	1.32895
4	…	…	1.32480

$\because |x_4 - x_3| < \varepsilon$

$\therefore x^* \approx x_4 = 1.32480$

§3 牛顿法及其变形

一 牛顿法

1 基本思想：非线性问题线性化,以直代曲

2 公式：设 x_k是$f(x)=0$的近似解，将$f(x)$在x_k处Taylor展开：

$$f(x)=f(x_k)+f'(x_k)(x-x_k)+o((x-x_k)^2)$$

则$f(x_k)+f'(x_k)(x-x_k)=0$为$f(x)=0$的近似表达式；

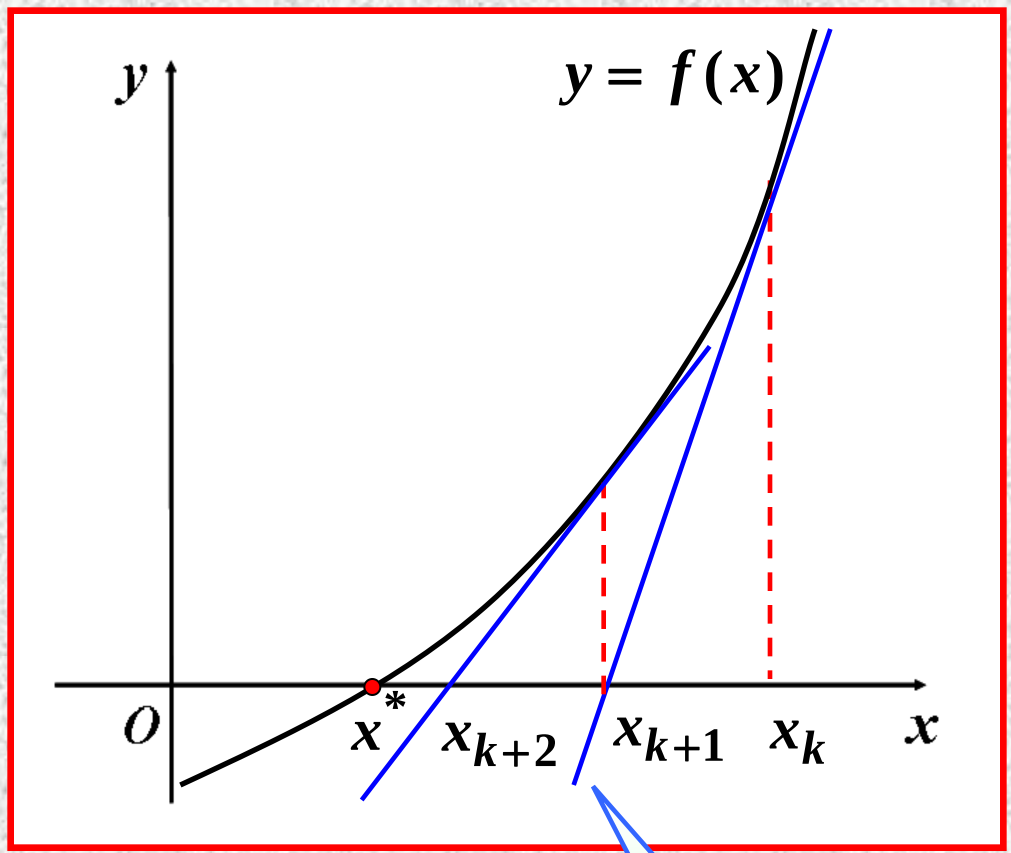
其解为：
$$x=x_k-\frac{f(x_k)}{f'(x_k)}$$

将x作为$f(x)=0$新的近似解，记作x_{k+1}.

即：
$$\boxed{x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)},\quad \text{其中} f'(x_k)\neq 0;\quad k=0,1,\cdots}$$

3 几何意义： 在x^*附近用过点$(x_k, f(x_k))$的切线近似代替 $f(x)$，

而切线与x轴的交点为 x_{k+1}.



切线：$l_k(x) = f(x_k) + f'(x_k)(x - x_k)$

2. 收敛性

$\because$ 迭代格式 $\quad x_{k+1}=x_k-\dfrac{f(x_k)}{f'(x_k)}$

$\therefore$ 迭代函数 $\quad \varphi(x)=x-\dfrac{f(x)}{f'(x)}$

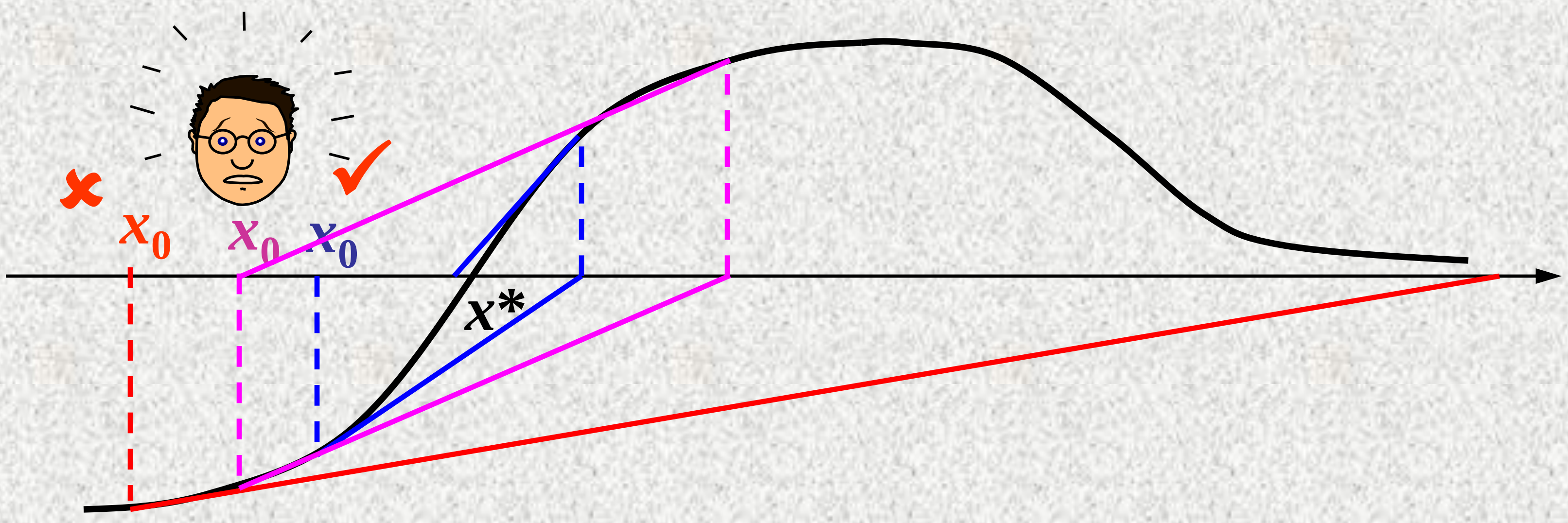
设x^*为$f(x)=0$的单根,即 $f(x^*)=0,\ f'(x^*)\neq 0,$

$$\therefore \varphi'(x)=1-\frac{[f'(x)]^2-f(x)f''(x)}{[f'(x)]^2}=\frac{f(x)f''(x)}{[f'(x)]^2}$$

$\therefore \varphi'(x^*)=0,\quad$ 而$\varphi''(x^*)=\dfrac{f''(x^*)}{f'(x^*)}\neq 0$(一般)

?

故牛顿法至少是$\boxed{\text{局部}}$平方收敛的.



注：Newton’s Method 收敛性依赖于x_0 的选取。

例1 用牛顿法求$\sqrt{115}$的近似值($\varepsilon = 0.00001$).

解:求$\sqrt{115} \Leftrightarrow$ 求$x = \sqrt{115}$

$\Leftrightarrow$ 求$x^2 - 115 = 0$的正根.

$$x_{k+1} = x_k - \frac{x_k^2 - 115}{2x_k} = \frac{1}{2}\left(x_k + \frac{115}{x_k}\right)$$

取$x_0 = 10$,得: $x_1 = 10.75$

$x_2 = 10.723837$

$x_3 = 10.723805$

$x_4 = 10.723805$

$\because |x_4 - x_3| < \varepsilon \quad \therefore x^* \approx x_4 = 10.723805$

二 牛顿法的变形

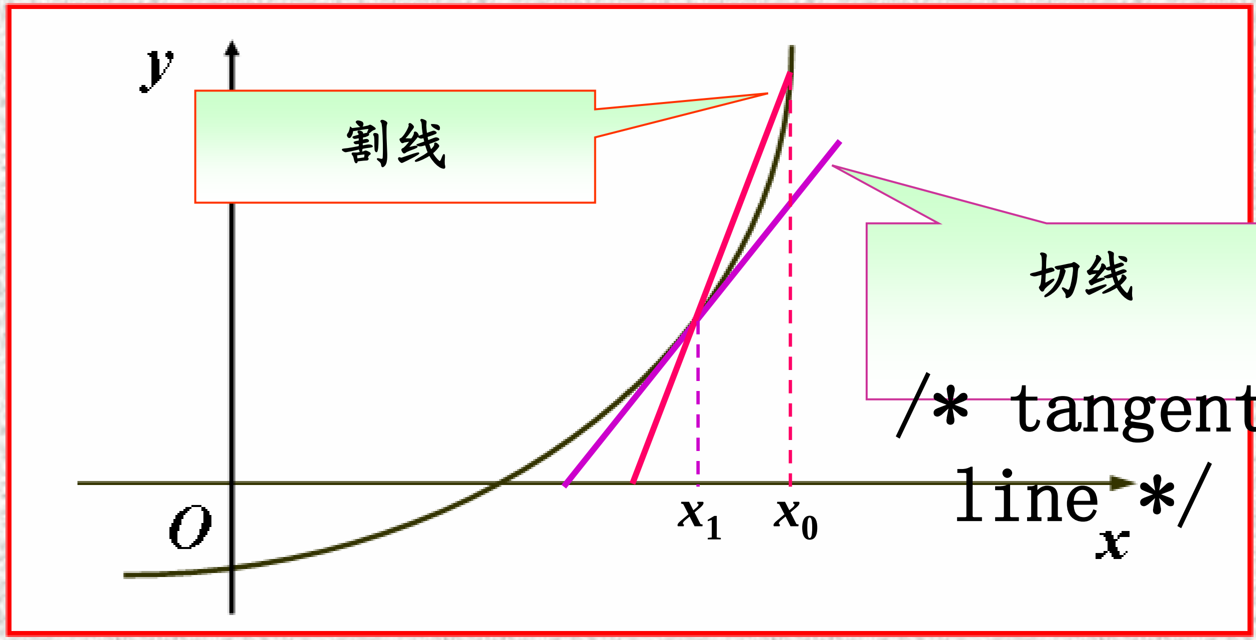
注：1. 割线法的收敛阶数为$p=\frac{1}{2}(1+\sqrt{5})\approx 1.618$
局部收敛.
2. 割线法需要两个初始值 x_0, x_1.

1 简化牛顿法

$$取 f'(x_k)=M, 则 x_{k+1}=x_k-\frac{f(x_k)}{M}, \quad k=0,1,2,\cdots$$

2 割线法

$$取 f'(x_k)\approx\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}, 则 x_{k+1}=x_k-\frac{f(x_k)(x_k-x_{k-1})}{f(x_k)-f(x_{k-1})}$$



三 计算重根的牛顿迭代法

1 问题

若x^*为$f(x)=0$的$m(m>1)$重根，则$f(x)$可表达为

$$f(x)=(x-x^*)^m g(x),\quad \text{其中} g(x^*)\neq 0$$

考察用牛顿法求x^*的收敛速度.

2 结论：方法收敛，但收敛速度下降

考察$x_{k+1}=x_k-\dfrac{f(x_k)}{f'(x_k)}$的收敛速度：

$$\boxed{f(x)=(x-x^*)^m g(x)}$$

$$\because e_{k+1}=x_{k+1}-x^*=x_k-x^*+x_{k+1}-x_k$$

$$=e_k-\frac{(x_k-x^*)^m g(x_k)}{m(x_k-x^*)^{m-1}g(x_k)+(x_k-x^*)^m g'(x_k)}$$

$$=e_k-\frac{(x_k-x^*)g(x_k)}{mg(x_k)+(x_k-x^*)g'(x_k)}$$

$$=e_k-\frac{e_k g(x_k)}{mg(x_k)+e_k g'(x_k)} \triangleq e_k B(x_k)$$

$$\therefore \lim_{k\to\infty}\frac{e_{k+1}}{e_k}=\lim_{k\to\infty}B(x_k)=1-\frac{1}{m}\neq 0$$

$\therefore$此时仅为局部线性收敛，且重数m越大收敛越慢!

3 计算重根加速方法

✍方案1: **化重根为单根**

令$u(x)=\dfrac{f(x)}{f'(x)}=\dfrac{(x-x^*)\boxed{g(x)}}{\boxed{mg(x)+e_k g'(x)}}\triangleq(x-x^*)Q(x)$,

则$Q(x^*)=\dfrac{1}{m}\neq 0$, 即$x^*$为$u(x)$的单根.

因此对$u(x)$利用牛顿法，则为局部平方收敛.

✍方案2:取迭代格式 $x_{k+1} = x_k - m\dfrac{f(x_k)}{f'(x_k)}$

- 可以证明是局部平方收敛的迭代格式。
- 如何确定重根数m: $\boxed{\text{边迭代边估计的方法}}$

$$\lambda_k = \frac{x_k - x_{k-1}}{x_{k-1} - x_{k-2}} = \frac{(x_k - x^*) - (x_{k-1} - x^*)}{(x_{k-1} - x^*) - (x_{k-2} - x^*)} = \frac{e_k - e_{k-1}}{e_{k-1} - e_{k-2}}$$

$$\therefore \lim_{k\to\infty} \lambda_k = \lim_{k\to\infty} \frac{e_k - e_{k-1}}{e_{k-1} - e_{k-2}} = \lim_{k\to\infty} \frac{e_k}{e_{k-1}} \cdot \frac{1 - \dfrac{e_{k-1}}{e_k}}{1 - \dfrac{e_{k-2}}{e_{k-1}}} = \lim_{k\to\infty} \frac{e_k}{e_{k-1}} = 1 - \frac{1}{m}$$

$$\therefore \lambda_k = 1 - \frac{1}{m},\ m = \frac{1}{1 - \lambda_k}$$

【评注】

Newton法是一种求解$f(x)=0$行之有效的迭代法,在单根附近具有较高的收敛性。但不足之处在于:

1）关键在于选取足够精确的迭代初值，若选择不当，可能发散;

2）另一局限在于计算导数$f`(x)$，若$f(x)$的形式复杂不便求之，则利用估计值差商代替，但收敛速度下降。

练习

1、已知方程$f(x)=x^3-2x-3$在区间[1,2]内有一根。

A：试问能否用迭代格式$x_{k+1}=0.5(x_k^3-3)$求此根？并说明理由。

B：选择一个收敛的迭代格式求此根，使其精度达到 0.1。

2、设$\varphi(x)=x+c(x^2-3)$，如何选取c，使得迭代格式

$x_{k+1}=\varphi(x_k)$具有局部收敛性。

3、给定方程$f(x)=x^3-12x+16=(x-2)^2(x+4)$

A：用牛顿法求其正根的近似值($x_0=1.95,\varepsilon=0.01$)；

B：讨论上述方法求正根的收敛性。

4、证明：$x_{k+1}=x_k-m\dfrac{f(x_k)}{f'(x_k)}$是求$f(x)=0$的$m$重根的具有平方收敛的迭代格式。

§ 4非线性方程组的求解方法

□ 问题

求解 $\begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) = 0 \end{cases}$，其中$n > 1$, f_i：$D \subset R^n \to R$。

若记 $\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\boldsymbol{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

则等价为 $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}$

其中$\boldsymbol{F}$：$D \subset R^n \to R^n$为非线性映射（向量值函数）。

例如，求 $\begin{cases} f_1(x_1, x_2) = \frac{1}{2} x_1 \sin(\frac{1}{2}\pi x_1) - x_2 = 0 \\ f_2(x_1, x_2) = x_2^2 - x_1 + 1 = 0 \end{cases}$ 的根。

一 预备知识:多元函数Taylor展开

设$f:R^2\to R^2$，$x=(x_1,x_2)^T$，$x^0=(x_1^0,x_2^0)^T$,则

$$f_1(x)=f_1(x^0)+\left(\frac{\partial f_1}{\partial x_1}(x_1-x_1^0)+\frac{\partial f_1}{\partial x_2}(x_2-x_2^0)\right)\Bigg|_{x=x^0}+o(\rho)$$

$$=f_1(x^0)+f_1'(x^0)(x-x^0)+o(\rho)$$

$$f_2(x)=f_2(x^0)+\left(\frac{\partial f_2}{\partial x_1}(x_1-x_1^0)+\frac{\partial f_2}{\partial x_2}(x_2-x_2^0)\right)\Bigg|_{x=x^0}+o(\rho)$$

$$=f_2(x^0)+f_2'(x^0)(x-x^0)+o(\rho)$$

即(矩阵形式)：$f=\begin{pmatrix}f_1(x)\\ f_2(x)\end{pmatrix}=\begin{pmatrix}f_1(x^0)\\ f_2(x^0)\end{pmatrix}+\begin{pmatrix}\frac{\partial f_1}{\partial x_1},\frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1},\frac{\partial f_2}{\partial x_2}\end{pmatrix}\begin{pmatrix}x_1-x_1^0\\ x_2-x_2^0\end{pmatrix}+o(\rho)$

$$\approx f(x^0)+f'(x^0)(x-x^0)$$

其中,$(x-x^0)=\begin{pmatrix}x_1-x_1^0\\ x_2-x_2^0\end{pmatrix}$,$\rho=\sqrt{\Delta x_1^2+\Delta x_2^2}$, $f'(x^0)=\begin{pmatrix}f_1'(x^0)\\ f_2'(x^0)\end{pmatrix}=\begin{pmatrix}\frac{\partial f_1}{\partial x_1},\frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1},\frac{\partial f_2}{\partial x_2}\end{pmatrix}$ (f的Jacobi矩阵)

如$\begin{cases}f_1(x_1,x_2)=x_1^2-2x_1-x_2+0.5=0\\ f_2(x_1,x_2)=x_1^2+4x_2^2-4=0\end{cases}$

在$x^0=(1,1)^T$的Taylor展开（近似表达式）：

$$f(x)\approx f(x^0)+\begin{pmatrix}2x_1-2 & -1\\ 2x_1 & 8x_2\end{pmatrix}\Bigg|_{(1,1)^T}(x-x^0)$$

$$=\begin{pmatrix}-1.5\\ 1\end{pmatrix}+\begin{pmatrix}0 & -1\\ 2 & 8\end{pmatrix}\begin{pmatrix}x_1-1\\ x_2-1\end{pmatrix}$$

□ 推广：n元函数情形

设$\boldsymbol{F}: D \subset R^n \to R^n$，其一阶Taylor展式为

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}^0) + \boldsymbol{F}'(\boldsymbol{x}^0)(\boldsymbol{x} - \boldsymbol{x}^0) + o(\|\boldsymbol{x} - \boldsymbol{x}^0\|)$$

其中，$\boldsymbol{F}'(\boldsymbol{x}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}$(称之为$F$的Jacobi矩阵)

二 不动点迭代法

◆ 基本思想 转化为非线性映射的不动点

$$F(x)=0 \xleftrightarrow{\text{等价变形}} x=G(x)$$

如果$\exists x^*$,使$x^*=G(x^*)$,则称x^*为$\varphi(x)$的一个不动点.

◆ 不动点迭代算法

取初值：$x^0\in R^n$，令$x^{k+1}=G(x^k)\quad (k=0,1,2,\cdots)$

如, 求解$\begin{cases} f_1(x,y)=x^2-2x-y+0.5=0 \\ f_2(x,y)=x^2+4y^2-4=0 \end{cases} \Leftrightarrow \begin{cases} x=\dfrac{x^2-y+0.5}{2} \\ y=\dfrac{-x^2-4y^2+8y+4}{8} \end{cases}$

◆ 收敛速度

设 $\lim\limits_{k\to\infty} x^k = x^*$, 如果 $\lim\limits_{k\to\infty}\dfrac{\left\|x^{k+1}-x^*\right\|}{\left\|x^k-x^*\right\|^p} = c \neq 0$, 则称迭代法是p阶收敛的.

当$p=1$时，称为线性收敛；

当$p=2$时，称为平方收敛；

当$p>1$时，称为超线性收敛.

◆ 压缩映射

➢ 定义

设$G: D\subset R^n \to R^n$, 若∃常数$0<L<1$, 使得对$\forall x, y\in D_0\subset D$, 有

$$\|G(x)-G(y)\|\le L\|x-y\|$$

则称G在D_0上是压缩映射.

➢压缩映射原理

设$\boldsymbol{G}: D \subset R^n \to R^n$为闭集$D_0 \subset D$上的压缩映射，且$\boldsymbol{G}(D_0) \subset D_0$，则$G$在$D_0$上存在唯一的不动点，且对$\boldsymbol{x}^0 \in D_0$，由迭代法

$$\boldsymbol{x}^{k+1} = \boldsymbol{G}(\boldsymbol{x}^k) \quad (k = 0, 1, 2, \cdots)$$

生成的$\left\{\boldsymbol{x}^k\right\}$都收敛于$\boldsymbol{x}^*$，且有如下误差估计

$$\left\|\boldsymbol{x}^* - \boldsymbol{x}^k\right\| \le \frac{L}{1-L}\left\|\boldsymbol{x}^k - \boldsymbol{x}^{k-1}\right\|, \quad \left\|\boldsymbol{x}^* - \boldsymbol{x}^k\right\| \le \frac{L^k}{1-L}\left\|\boldsymbol{x}^1 - \boldsymbol{x}^0\right\|$$

➢局部判定定理

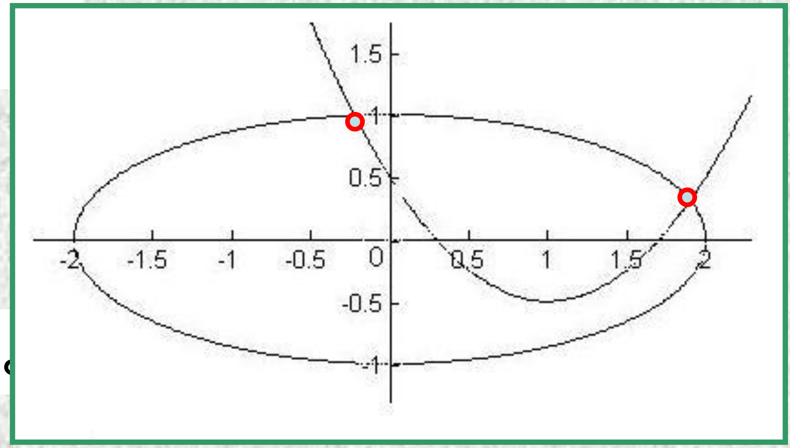
设$\boldsymbol{G}: D \subset R^n \to R^n$有不动点$\boldsymbol{x}^* \in \mathrm{int}(D)$，且$G$在$\boldsymbol{x}^*$有一阶连续偏导数，如果$\boldsymbol{G}'(\boldsymbol{x}^*)$的谱半径$\rho\left(\boldsymbol{G}'(\boldsymbol{x}^*)\right) < 1$，则存在开球

$$S = \left\{\boldsymbol{x} \in R^n \,\middle|\, \left\|\boldsymbol{x} - \boldsymbol{x}^*\right\| < \delta\right\} \subset D$$

使得对$\forall \boldsymbol{x}^0 \in S$，由迭代生成的序列$\left\{\boldsymbol{x}^k\right\}$收敛于$\boldsymbol{x}^*$。

例 1 求解 $\begin{cases} f_1(x_1,x_2)=x_1^2-2x_1-x_2+0.5=0 \\ f_2(x_1,x_2)=x_1^2+4x_2^2-4=0 \end{cases}$

解 用Matlab函数ezplot作图, 可见



$\boldsymbol{x}^{(0)}=(-0.2,1)^T$ 和 $\tilde{\boldsymbol{x}}^{(0)}=(2,0.3)^T$ 附近有根。

原方程 $\Leftrightarrow \begin{cases} x_1=g_1(x_1,x_2)=\dfrac{x_1^2-x_2+0.5}{2} \\ x_2=g_2(x_1,x_2)=\dfrac{-x_1^2-4x_2^2+8x_2+4}{8} \end{cases}$

初值分别选为: $\boldsymbol{x}^{(0)}=(-0.2,1)^T,\ \tilde{\boldsymbol{x}}^{(0)}=(2,0.3)^T$

迭代结果

迭代初值	$(x_1^0, x_2^0)^T = (-0.2, 1)^T$		$(x_1^0, x_2^0)^T = (2, 0.3)^T$	
	$x_1^{(k)}$	$x_2^{(k)}$	$x_1^{(k)}$	$x_2^{(k)}$
1	-0.2300000	0.9950000	2.1000000	0.255000
2	-0.2210500	0.9933750	2.3275000	0.171237 0 2
3	-0.2222559	0.9938702	2.8730094	5 0.020580
4	-0.2222362	0.9938065	4.3873818 10.150842	0.552565 7
5	-0.2222088	0.9938072	53.075481 1	2.611369 1 3
6	-0.2222152	0.9938087	1417.9538 9	18.40094 4
7	-0.2222146	0.9938084	618	539.3241 37
	收 敛		发 散	554

$$\begin{cases} f_1(x_1,x_2)=x_1^2-2x_1-x_2+0.5=0 \\ f_2(x_1,x_2)=x_1^2+4x_2^2-4=0 \end{cases}$$

➢理论分析

$$\because \begin{cases} x_1=g_1(x_1,x_2)=\dfrac{x_1^2-x_2+0.5}{2} \\ x_2=g_2(x_1,x_2)=\dfrac{-x_1^2-4x_2^2+8x_2+4}{8} \end{cases}$$

即：$x=G(x)$

所以，映射G的Jacobi矩阵

$$\boldsymbol{G}'(\boldsymbol{x})=\begin{pmatrix} \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_1}{\partial x_2} \\ \dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_2}{\partial x_2} \end{pmatrix}=\begin{pmatrix} x_1 & -1/2 \\ -x_1/4 & -x_2+1 \end{pmatrix}$$

$\boldsymbol{x}^{(0)}=(-0.2,1)^T$, $\because \rho\left(\boldsymbol{G}'(\boldsymbol{x}^{(0)})\right)=0.158<1$ $\therefore$ 收敛

$\tilde{\boldsymbol{x}}^{(0)}=(2,0.3)^T$, $\because \rho\left(\boldsymbol{G}'(\tilde{\boldsymbol{x}}^{(0)})\right)=2.17>1$ $\therefore$ 发散

三 Newton迭代法

➢迭代公式

设$\boldsymbol{x}^k$是$\boldsymbol{F}(\boldsymbol{x})=0$的一个近似解，则

$$\boldsymbol{F}(\boldsymbol{x})=\boldsymbol{F}(\boldsymbol{x}^k)+\boldsymbol{F}'(\boldsymbol{x}^k)(\boldsymbol{x}-\boldsymbol{x}^k)+o(\rho)$$

因此$\boldsymbol{F}(\boldsymbol{x}^k)+\boldsymbol{F}'(\boldsymbol{x}^k)(\boldsymbol{x}-\boldsymbol{x}^k)=0$为$\boldsymbol{F}(\boldsymbol{x})=0$的近似表达式，

若$\boldsymbol{F}'(\boldsymbol{x}^k)$可逆，则 $\boldsymbol{x}=\boldsymbol{x}^k-[\boldsymbol{F}'(\boldsymbol{x}^k)]^{-1}\boldsymbol{F}(\boldsymbol{x}^k)$

记为：

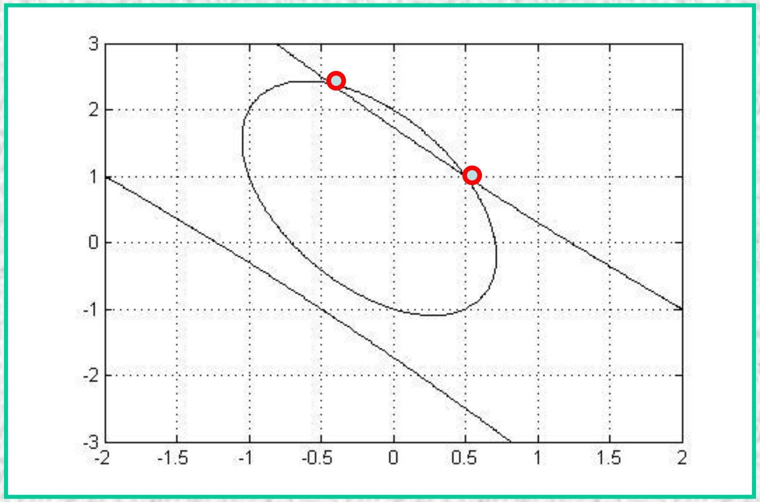
$$\boxed{\boldsymbol{x}^{k+1}=\boldsymbol{x}^k-[\boldsymbol{F}'(\boldsymbol{x}^k)]^{-1}\boldsymbol{F}(\boldsymbol{x}^k)}$$

——求解非性方程组的Newton迭代法

例2 用Newton迭代法求解非性方程组

$$\begin{cases} f_1(x_1,x_2)=4x_1^2+x_2^2+2x_1x_2-x_2-2=0 \\ f_2(x_1,x_2)=2x_1^2+3x_1x_2+x_2^2-3=0 \end{cases}$$

解 用Matlab函数ezplot作图



$$F(x)=\begin{bmatrix} 4x_1^2+x_2^2+2x_1x_2-x_2-2 \\ 2x_1^2+3x_1x_2+x_2^2-3 \end{bmatrix}$$

$$F'(x)=\begin{bmatrix} 8x_1+2x_2 & 2x_1+2x_2-1 \\ 4x_1+3x_2 & 3x_1+2x_2 \end{bmatrix}$$

$$x^{k+1}=x^k-[F'(x^k)]^{-1}F(x^k)$$

取初值$x^0=(0.4,0.9)^T$，$x^3=\begin{bmatrix}0.5000\\1.0000\end{bmatrix}$，$F(x^3)=\begin{bmatrix}0.9282\times10^{-7}\\0.0738\times10^{-7}\end{bmatrix}$

取初值$x^0=(-0.5,2.5)^T$，$x^3=\begin{bmatrix}-0.4277\\2.3868\end{bmatrix}$，$F(x^3)=\begin{bmatrix}0.2248\times10^{-8}\\-0.0173\times10^{-8}\end{bmatrix}$

➢局部收敛定理

设$\boldsymbol{F}: D\subset R^n \to R^n$，$\boldsymbol{x}^*$满足$\boldsymbol{F}(\boldsymbol{x}^*)=\boldsymbol{0}$，$\boldsymbol{F}$在$\boldsymbol{x}^*$的开邻域$S_0\subset D$上可导，且$\boldsymbol{F}'(\boldsymbol{x})$连续，$\boldsymbol{F}'(\boldsymbol{x}^*)$非奇异，则存在闭球$S\subset S_0$，使得映射$\boldsymbol{\Phi}(\boldsymbol{x})=\boldsymbol{x}-\boldsymbol{F}'(\boldsymbol{x})^{-1}\boldsymbol{F}(\boldsymbol{x})$有意义，且**Newton**迭代序列超线性收敛于$\boldsymbol{x}^*$，若还存在常数$L>0$，使得

$$\left\|\boldsymbol{F}'(\boldsymbol{x})-\boldsymbol{F}'(\boldsymbol{x}^*)\right\|\le L\left\|\boldsymbol{x}-\boldsymbol{x}^*\right\|,\forall \boldsymbol{x}\in S$$

则**Newton**迭代序列至少平方收敛于$\boldsymbol{x}^*$。