第八讲

复合函数求导的例

对数求导法

基本求导公式

例5 求函数 $y = \sin x^2$ 的导数 y'.

解 将 $y = \sin x^2$ 分解成 $y = \sin u$ 与 $u = x^2$ 这两个基本初等函数的复合， 于是由链式法则, 有

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = (\sin u)'(x^2)'$$

$$= \cos u \cdot 2x$$

$$= 2x\cos x^2.$$

例6 求幂函数 $y = x^{\alpha}$ (α是实数，$x > 0$) 的导数.

解 $y = x^{\alpha} = \mathrm{e}^{\alpha\ln x}$ 是由 $y = \mathrm{e}^{u}$ 与 $u = \alpha\ln x$ 复合而成,

故

$$(x^{\alpha})' = (\mathrm{e}^{\alpha\ln x})' = \mathrm{e}^{\alpha\ln x}\cdot(\alpha\ln x)'$$

$$= \mathrm{e}^{\alpha\ln x}\cdot\frac{\alpha}{x}$$

$$= \alpha x^{\alpha-1}.$$

例7 求下列函数的导数：

(i) $\sqrt{1+x^2}$； (ii) $\dfrac{1}{\sqrt{1+x^2}}$； (iii) $\ln(x+\sqrt{1+x^2})$.

解 运用复合求导法则，分别计算如下：

(i) $(\sqrt{1+x^2})' = \dfrac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot \left(1+x^2\right)' = \dfrac{x}{\sqrt{1+x^2}}$.

(ii) $\left(\dfrac{1}{\sqrt{1+x^2}}\right)' = -\dfrac{1}{2}(1+x^2)^{-3/2} \cdot (1+x^2)'$

$$= -\frac{x}{\sqrt{(1+x^2)^3}}.$$

(iii) $$\left[\ln(x+\sqrt{1+x^2})\right]'$$

$$=\frac{1}{x+\sqrt{1+x^2}}\left(x+\sqrt{1+x^2}\right)'$$

$$=\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$=\frac{1}{x+\sqrt{1+x^2}}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)=\frac{1}{\sqrt{1+x^2}}.$$

例8 求下列函数的导数:

$$\sin^2\frac{x}{2}=\frac{1}{2}(1-\cos x)$$
$$\cos^2\frac{x}{2}=\frac{1}{2}(1+\cos x)$$

(i) $f(x)=\dfrac{2}{3}\arctan\left(\dfrac{1}{3}\tan\dfrac{x}{2}\right)$;

解 (i) $$f'(x)=\frac{2}{3}\cdot\frac{1}{1+\frac{1}{9}\tan^2\frac{x}{2}}\cdot\frac{1}{3}\sec^2\frac{x}{2}\cdot\frac{1}{2}$$

$$=\frac{2}{3}\frac{9}{9\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}\frac{1}{6}\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}.$$

$$=\frac{1}{9\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{1}{5+4\cos x}.$$

(ii) $g(x)=\begin{cases}\dfrac{x}{1+e^{1/x}}, & x\neq 0,\\ 0, & x=0.\end{cases}$

(ii) 当 $x\neq 0$ 时, $g'(x)=\dfrac{1+e^{1/x}+\dfrac{1}{x}e^{1/x}}{(1+e^{1/x})^2}$.

当 $x=0$ 时, 因为

$$g'_{+}(0)=\lim_{\Delta x\to 0^+}\frac{1}{\Delta x}\left(\frac{\Delta x}{1+e^{1/\Delta x}}-0\right)=0,$$

$$g'_{-}(0)=\lim_{\Delta x\to 0^-}\frac{1}{\Delta x}\left(\frac{\Delta x}{1+e^{1/\Delta x}}-0\right)=1,$$

所以 g 在 $x=0$ 处不可导.

对幂指函数 $u(x)^{v(x)}$，可用对数求导法求导.

对数求导法：设 $u(x)>0$, $u(x)$与$v(x)$均可导，则

$$(u(x)^{v(x)})' = (\mathrm{e}^{v(x)\ln u(x)})' = \mathrm{e}^{v(x)\ln u(x)}(v(x)\ln u(x))'$$

$$= u(x)^{v(x)}\left[v'(x)\ln u(x) + v(x)\,\frac{u'(x)}{u(x)}\right].$$

例如，对于函数 $y = x^{\sin x}$,

$$\left(x^{\sin x}\right)' = x^{\sin x}\left[\cos x\ln x + \sin x\,\frac{1}{x}\right]$$

对数求导法不仅对幂指函数 $u(x)^{v(x)}$ 有效, 也能简化某些连乘、连除式的求导.

例9 设 $y=\dfrac{(x^2+1)^3(x-2)^{1/4}}{(5x-9)^{2/5}}$, 求 y'.

解 先对函数两边取对数，得

$$\ln y=3\ln(x^2+1)+\frac{1}{4}\ln(x-2)-\frac{2}{5}\ln(5x-9).$$

再对上式两边求 x 导数，得到

$$\frac{y'}{y}=\frac{6x}{x^2+1}+\frac{1}{4(x-2)}-\frac{2}{5}\cdot\frac{5}{5x-9}.$$

于是

$$y'=\frac{(x^2+1)^3(x-2)^{1/4}}{(5x-9)^{2/5}}\left[\frac{6x}{x^2+1}+\frac{1}{4(x-2)}-\frac{2}{5x-9}\right].$$

基本求导法则与公式

求导法则:

(1) $(u \pm v)' = u' \pm v';$

(2) $(uv)' = u'v + uv', (cu)' = cu'$ (c 为常数);

(3) $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}, \left(\dfrac{1}{v}\right)' = -\dfrac{v'}{v^2};$

(4) 反函数的导数 $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}};$

(5) 复合函数的导数 $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}.$

基本初等函数的导数公式:

(1) $(c)' = 0$ (c为常数);

(2) $(x^{\alpha})' = \alpha x^{\alpha-1}$ (α为任意实数);

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}};\quad (x^{-1})' = -x^{-2} = -\frac{1}{x^2};$$

$$\left(\frac{1}{\sqrt{x}}\right)' = (x^{-\frac{1}{2}})' = -\frac{1}{2x^{\frac{3}{2}}};$$

(3) $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$;

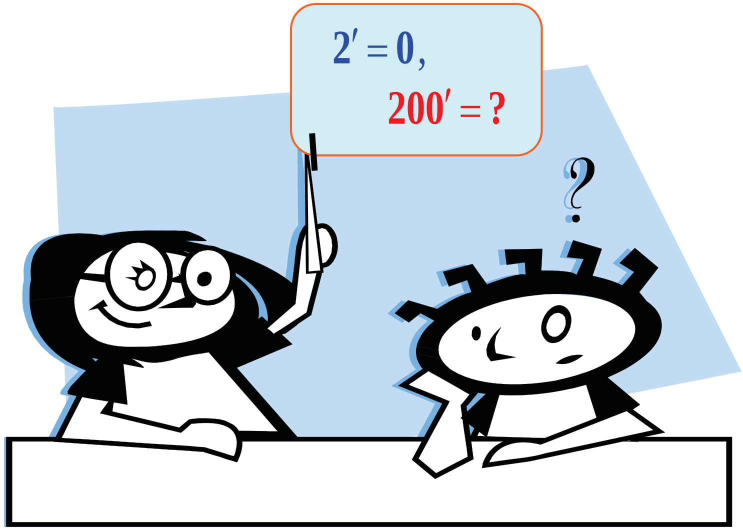
(4) $(\tan x)' = \sec^2 x$, $(\cot x)' = -\csc^2 x$;

$(\sec x)' = \sec x \tan x$, $(\csc x)' = -\csc x \cot x$;

(5) $(a^x)' = a^x \cdot \ln a$, $(\mathrm{e}^x)' = \mathrm{e}^x$;

(6) $(\log_a x)' = \dfrac{1}{x\ln a}$, $(\ln x)' = \dfrac{1}{x}$;

(7) $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$, $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$,



$(\arctan x)' = \dfrac{1}{1+x^2}$,

$(\operatorname{arccot} x)' = -\dfrac{1}{1+x^2}$.